Spherical Trigonometry

Terrametra Resources

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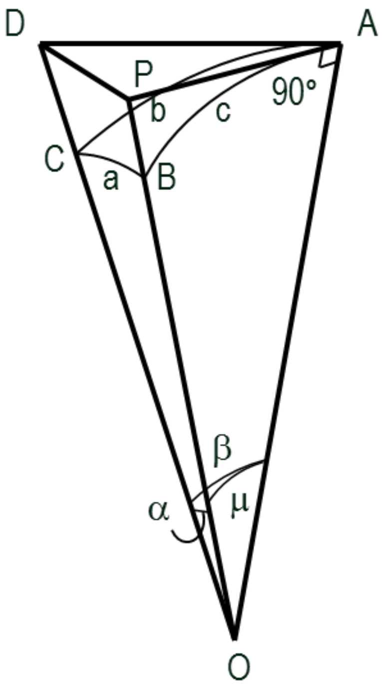
The Spherical Trig *"LAWS"*

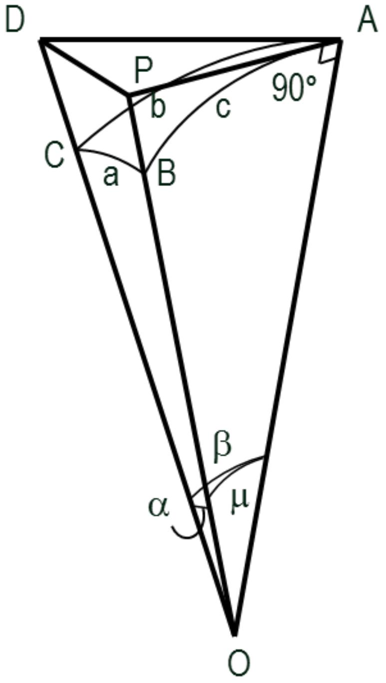
- ***Law of Cosines***
- ***Law of Sines***
- ***Law of Cotangents***
- ***Spherical Excess***



Basic Assumptions

Sides AD and AP are tangent at A such that …

- DAP = spherical angle A
- DAO = 90°
- PAO = 90°



Plane Trig Formulae

In plane triangle DPO …

- **Equation 1**

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos\alpha$$

In plane triangle DPA …

- **Equation 2**

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

In plane triangle ADO …

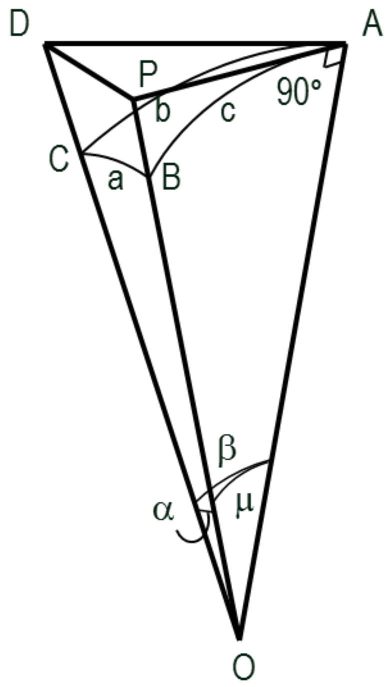
- **Equation 3**

$$DO^2 = AD^2 + AO^2$$

In plane triangle APO …

- **Equation 4**

$$PO^2 = AP^2 + AO^2$$

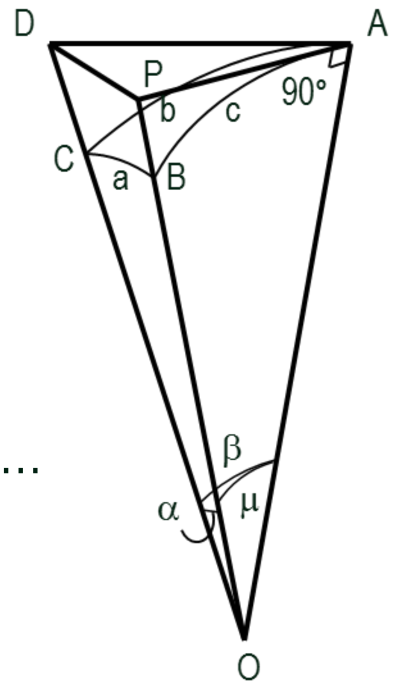


- **Equation 1**

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos\alpha$$

- **Equation 2**

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$



Equate the right sides of equations 1 and 2 and rearrange …

$$DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos\alpha = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

- **Equation 5**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

- **Equation 3**

$$DO^2 = AD^2 + AO^2$$

- **Equation 4**

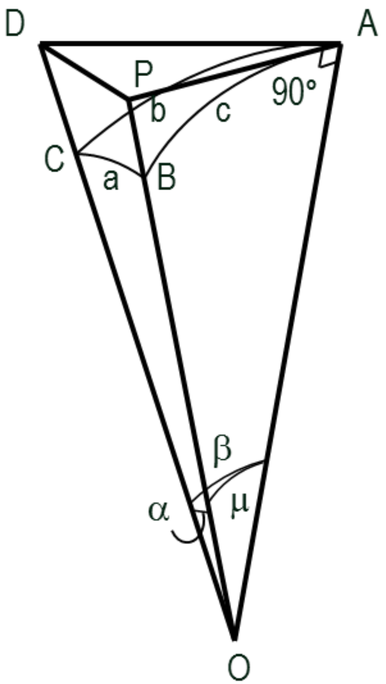
$$PO^2 = AP^2 + AO^2$$

Add equations 3 and 4 and rearrange …

$$DO^2 + PO^2 = AD^2 + AP^2 + AO^2 + AO^2$$

- **Equation 6**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$

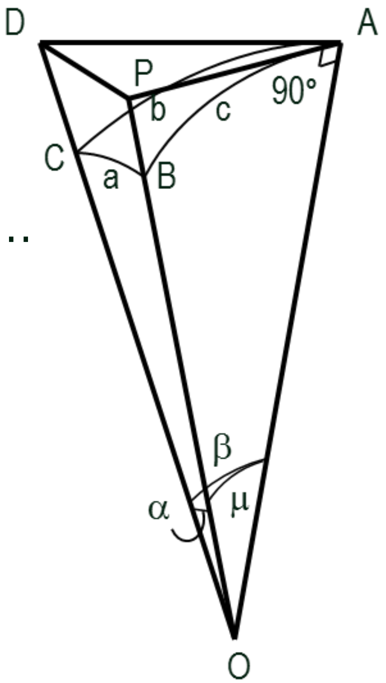


Equation 5

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

Equation 6

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$



Equate the right sides of equations 5 and 6 and rearrange …

$$2 \cdot DO \cdot PO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A = 2 \cdot AO^2$$

$$DO \cdot PO \cdot \cos\alpha = AO \cdot AO + AD \cdot AP \cdot \cos A$$

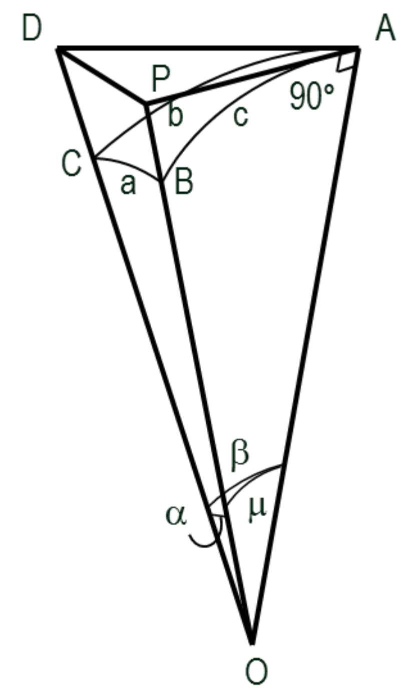
$$\cos\alpha = \frac{AO \cdot AO}{DO \cdot PO} + \frac{AD \cdot AP}{DO \cdot PO}\cos A$$

$$\cos\alpha = \cos\beta \cdot \cos\mu + \sin\beta \cdot \sin\mu \cdot \cos A$$

"LAW of COSINES"

$$\cos\alpha = \cos\beta \cdot \cos\mu + \sin\beta \cdot \sin\mu \cdot \cos A$$

Note … $\alpha = a \quad \beta = b \quad \mu = c$



"LAW of COSINES"

- **Equation 7**

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

- **Equation 8**

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

- **Equation 9**

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$

Rearrange equations 7, 8 and 9 …

- **Equation 10** (from equation 7)

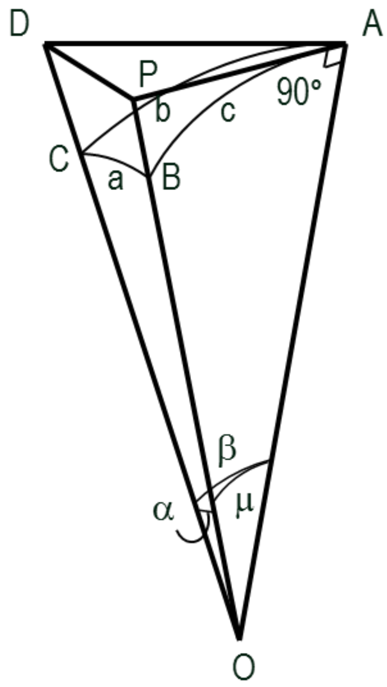
$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

- **Equation 11** (from equation 8)

$$\cos b - \cos a \cdot \cos c = \sin a \cdot \sin c \cdot \cos B$$

- **Equation 12** (from equation 9)

$$\cos c - \cos a \cdot \cos b = \sin a \cdot \sin b \cdot \cos C$$



- **Equation 10** (from equation 7)

$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

Square both sides of equation 10

$$\cos^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \cos^2 b \cdot \cos^2 c = \sin^2 b \cdot \sin^2 c \cdot \cos^2 A$$

Apply the trig identity … $\cos^2 \theta = 1 - \sin^2 \theta$

$$(1 - \sin^2 a) - 2\ \cos a\ \cos b\ \cos c + (1 - \sin^2 b)(1 - \sin^2 c) = \sin^2 b\ \sin^2 c\ (1 - \sin^2 A)$$

$$1 - \sin^2 a - 2\ \cos a\ \cos b\ \cos c + 1 - \sin^2 b - \sin^2 c + \sin^2 b\ \sin^2 c =$$
$$\sin^2 b\ \sin^2 c - \sin^2 b\ \sin^2 c\ \sin^2 A$$

- **Equation 13** (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$

"LAW of SINES"

- **Equation 13** (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$

- **Equation 14** (from equation 11)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B$$

- **Equation 15** (from equation 12)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

Equate the right sides of equations 13, 14 and 15 and rearrange …

$$-\sin^2 b \cdot \sin^2 c \cdot \sin^2 A = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

$$\sin b \cdot \sin c \cdot \sin A = \sin a \cdot \sin c \cdot \sin B = \sin a \cdot \sin b \cdot \sin C$$

"LAW of SINES"

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

- **Equation 16** (Law of Sines)

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$

- **Equation 17** (Law of Cosines)

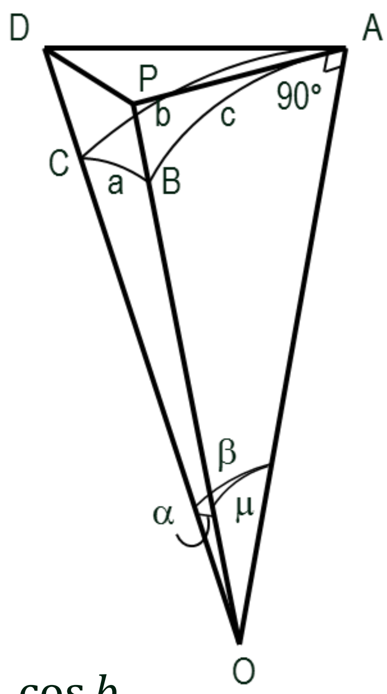
$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

- **Equation 18** (Law of Cosines)

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

- **Equation 19** (Law of Cosines)

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$



Substitute the right side of equation 18 into equation 17 for $\cos b$

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = \cos a \cdot \cos^2 c + \sin a \cdot \sin c \cdot \cos c \cdot \cos B + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a - \cos a \cdot \cos^2 c = \sin c \cdot \sin a \cdot \cos c \cdot \cos B + \sin c \cdot \sin b \cdot \cos A$$

$$\cos a \cdot (1 - \cos^2 c) = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

Apply the trig identity … $\sin^2 \theta = 1 - \cos^2 \theta$

$$\cos a \cdot \sin^2 c = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

$$\cos a \cdot \sin c = \sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A$$

- **Equation 20**

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$

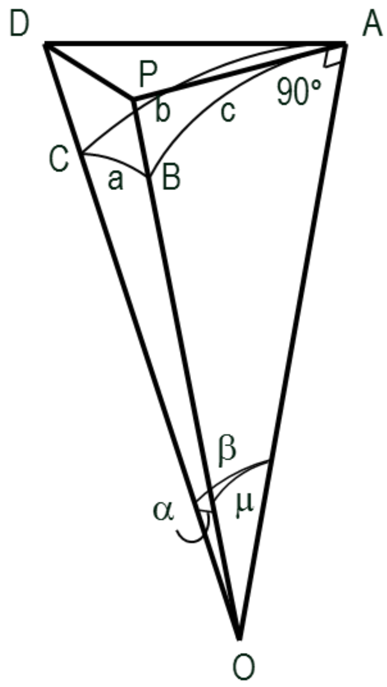
"LAW of COTANGENTS"

- **Equation 20**

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$

- **Equation 16**

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$



Divide equation 16 into equation 20 and rearrange …

$$\frac{\sin b \cdot \cos A}{\sin b \cdot \sin A} = \frac{\cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B}{\sin a \cdot \sin B}$$

"LAW of COTANGENTS"

$$\cot A = \frac{\cot a \cdot \sin c - \cos c \cdot \cos B}{\sin B}$$

… or …

$$\tan A = \frac{\sin B}{\cot a \cdot \sin c - \cos c \cdot \cos B}$$

"SPHERICAL EXCESS"

"SPHERICAL EXCESS"

$$E = A + B + C - 180°$$

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{s}{2}\right) \cdot \tan\left(\frac{s-a}{2}\right) \cdot \tan\left(\frac{s-b}{2}\right) \cdot \tan\left(\frac{s-c}{2}\right)}$$

… where …

$$s = \frac{a+b+c}{2}$$